

DMDE/M-19

ADVANCED ABSTRACT ALGEBRA

Paper-MM-401

Time : Three Hours]

[Maximum Marks : 80

Note : Attempt five questions in all, selecting at least one question from each unit. All questions carry equal marks.

UNIT-I

1. (a) Let $H \triangleleft G$. Prove that if G has a composition series, then G/H also has a composition series.
- (b) State and prove three subgroups Lemma of P. Hall.
2. (a) Let H be a proper subgroup of a nilpotent group G . Prove that $H \cap N_G(H) \neq H$.
- (b) Let $H \triangleleft G$. Prove that if both H and G/H are solvable, then so is G .

UNIT-II

3. (a) Prove that if P is a prime field, then either $P \cong \mathbb{Z}/p\mathbb{Z}$ or $P \cong \mathbb{Q}$.
- (b) Find the degree of the splitting field of the polynomial $(x^3 - 2)(x^2 - 4)$ over \mathbb{Q} .
4. (a) Prove that a finite normal extension is a splitting field.
- (b) Prove that if $\infty \in K$ is separable, then $F(\infty)/F$ is separable.
5. (a) Find the Galois group of $x^4 - 2$ over \mathbb{Q} .
- (b) Prove that the Galois group of $x^n - a \in F[x]$ is solvable.

UNIT-III

6. (a) Prove that if $T \in A_F(V)$ has all ch. roots in F , then T satisfies a polynomial of degree $n = \dim_F(V)$ over F .
- (b) Let $T \in A_F(V)$ be such that $T^m = I$. Prove that if $\text{ch}(F) = 0$, then T is diagonalizable. ($\dim_F(V) < \infty$).

7. (a) Prove that invariants of a nilpotent linear transformation are unique.
- (b) Prove that if $f(x) \in F[x]$, then the companion matrix $C(f(x))$ of $f(x)$ satisfies the polynomial $f(x)$.

UNIT-IV

8. (a) Let I be a left ideal of a ring R with unity. Prove that I is a direct summand of R iff I is generated by an idempotent.
- (b) Let R be a ring with unity. Prove that a left R -module is simple iff $M \cong R/I$ where I is a maximal left ideal of R .
9. (a) Prove that in an Artinian ring every nil left ideal is nilpotent.
- (b) Prove that each ideal in a Noetherian ring contains a finite product of prime ideals.
10. (a) Let M be a left module over a Noetherian ring. Prove that every non-zero submodule of M contains a uniform module.
- (b) Prove that over a PID, submodule of a finitely generated module is again finitely generated.